

A conjecture about the orthogonal vector equations associated with the Hadamard matrix

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Abstract. The investigation of Hadamard matrices provides an ancient problem in algebra which has extensive links to various real-world applications, such as signal transformation. In this work, we investigate a conjecture on the form of the solution corresponding to a special case of 6×6 complex Hadamard matrices. Establishing this conjecture will substantially influence the development of so-called non-H2-reducible Hadamard matrices. To build our case, we first establish two encompassing lemmas based on the geometric properties of unit modulus complex numbers with sum zero. Then, using these lemmas we establish the conjecture for various special cases. Finally, we take the general form of the problem and simply it down to a system of trigonometric equations. The numerical evidence presented give strong evidence of validity suggested a potential path for a complete proof.

Keywords: complex number; orthogonal vector; plane and analytic geometry.

1. Introduction

The examination of Hadamard matrices has been an ancient problem in algebra and seen considerable interest with a multitude of research in the last century [1]. As a function-valued unitary matrix with all constant modulus, Hadamard matrices have numerous applications and therefore are of considerable interest in signal processing transformations. In the fundamental sciences, they have a strong indication for arrangements of mutually unbiased bases in quantum mechanics. The existence of mutually unbiased bases is, in a mathematical sense, the same as the existence of interrelated Hadamard matrices. There has been substantial debate if there are four mutually unbiased bases in dimension six (note that it is generally believed the answer is "no") [2]. Three mutually unbiased bases in dimension six have been developed with analytic and numerical methods [3]. This methodology is subsequently extended in semidefinite algorithms [4]. During these developments, various proposals were made such as H2-reducibility and analytic characterization, this research is very recent and highlights recent work in regards to Hadamard matrices. [5].

Nonetheless, H2-reducible matrices are just a small class of Hadamard matrices, opening the door to conjecturing a classification of all four-parameter Hadamard matrices [6]. There have been extensive breakthroughs into these conjectures, yet the complete classification of 6×6 Hadamard matrices is still open.

In this paper, we concern ourselves with a particular substructure within a 6×6 complex Hadamard matrix and pose a conjecture regarding the permutation relation of its solutions. Certifying this conjecture would place a structural limitation that will be critical in understanding non-H2-reducible Hadamard matrices. Our approach to this conjecture is simple. In Section 2 we develop background material analyzing the geometric arrangements of three and four unit modulus complex numbers summing to zero. In Section 3 we state the main conjecture and then verify many particular cases. We then shift to proving the general case using a system of trigonometric equations later to provide additional evidence for the conjecture. We wrap things up in Section 4 with a jam-packed summary and ideas for future work.

2. Preparation: Geometric Lemmas of Complex Sums

In this section, we establish two fundamental lemmas concerning the sum of complex numbers of unit modulus. These lemmas, rooted in the geometry of the complex plane, will be instrumental in proving the main conjecture in the next section.

2.1. The Case of Three Unit Modulus Complex Numbers

2.1.1 Lemma 2.1

Assume a, b, c are three complex numbers, that their modular lengths are all 1, and that they satisfy the equality $a + b + c = 0$. Then, the set $\{a, b, c\}$ must be a rotation of the set $\{c_0, c_0\omega, c_0\omega^2\}$, where $|c_0| = 1$ and $\omega = e^{i\frac{2\pi}{3}}$. Geometrically, these three numbers form an equilateral triangle on the unit circle in the complex plane.

2.1.2 Proof

Let us divide both sides of the equation by c . We get $\frac{a}{c} + \frac{b}{c} + 1 = 0$. Let us set $\frac{a}{c} = e^{i\alpha}$ and $\frac{b}{c} = e^{i\beta}$, where $\alpha, \beta \in (0, 2\pi]$. By separating the real and imaginary parts, we obtain the system of equations:

$$\begin{cases} \cos(\alpha) + \cos(\beta) + 1 = 0 \\ \sin(\alpha) + \sin(\beta) = 0 \end{cases} \quad (1)$$

The second equation implies that $\alpha = 2\pi - \beta$ or $\alpha = \beta \pm \pi$. Only the former is compatible with the first equation, which leads to $2\cos\alpha + 1 = 0$. This yields two sets of solutions for (α, β) : $(2\pi/3, 4\pi/3)$ or $(4\pi/3, 2\pi/3)$.

These solutions correspond to $(\frac{a}{c}, \frac{b}{c}) = (\omega, \omega^2)$ or (ω^2, ω) . If we let $c = c_0$ for some complex number c_0 with $|c_0| = 1$, we can conclude that (a, b, c) is a permutation of $c_0(1, \omega, \omega^2)$, completing the proof.

2.2. The Case of Four Unit Modulus Complex Numbers

2.2.1 Lemma 2.2

Suppose that there are four complex numbers: a, b, c, d with unit modulus and they satisfy the equation $a + b + c + d = 0$. Then they must be paired up such that one pair are opposites of each other, and the other pair are also opposites. That is, they must satisfy $a + b = 0$ and $c + d = 0$, up to a permutation of the variables.

2.2.2 Proof

Geometrically, the condition that four vectors of length 1 add up to zero can be understood as them connecting end to end to form a closed quadrilateral with equal side lengths, which must be a rhombus. A rhombus has two pairs of parallel sides of equal length. In the complex plane, this parallelism means the corresponding complex numbers are either equal or opposite. Since the vectors must form a closed loop ($a + b + c + d = 0$), they cannot be equal, so they must be pairwise opposite. For instance, if a is parallel to c and b is parallel to d , we must have $a = -c$ and $b = -d$. The other pairings follow by permuting the letters.

This lemma can be trivially generalized: if four complex numbers have the same non-zero modulus and their sum is zero, they must also be pairwise opposite.

3. Research Results: Analysis of the Conjecture

3.1. The Main Conjecture and Its Symmetries

In this section, we mainly study the following conjecture.

3.1.1 Conjecture 3.1

Let us consider six complex numbers with modulus 1: x, a, b, c, d, f, g . They form a 3×2 submatrix of a larger structure:

$$M = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & x & a & b & c \\ 1 & x & 1 & d & f & g \end{pmatrix} \quad (2)$$

Assuming that the three row vectors of this matrix are pairwise orthogonal, we get the following three equations:

$$\begin{cases} 2 + x + a + b + c = 0 \\ 2 + x + d + f + g = 0 \\ 1 + x + \bar{x} + a\bar{d} + b\bar{f} + c\bar{g} = 0 \end{cases} \quad (3)$$

Then, the vector (a, b, c) must be a permutation of the vector (d, f, g) . That is, $\{a, b, c\} = \{d, f, g\}$ as sets.

Before proceeding with the proof, we note an important symmetry.

3.1.2 Theorem 3.2

If conjecture 3.1 is satisfied for x , then conjecture 3.1 is also satisfied for \bar{x} .

3.1.3 Proof

If we take the complex conjugate of the entire system of equations (3), we obtain:

$$\begin{cases} 2 + \bar{x} + \bar{a} + \bar{b} + \bar{c} = 0 \\ 2 + \bar{x} + \bar{d} + \bar{f} + \bar{g} = 0 \\ 1 + \bar{x} + x + \bar{a}d + \bar{b}f + \bar{c}g = 0 \end{cases} \quad (4)$$

This new system is identical in form to the original, with all variables replaced by their conjugates. If a solution (x, a, \dots, g) satisfies the conjecture (i.e., (a, b, c) is a permutation of (d, f, g)), then the conjugated solution $(\bar{x}, \bar{a}, \dots, \bar{g})$ will also satisfy the condition that $(\bar{a}, \bar{b}, \bar{c})$ is a permutation of $(\bar{d}, \bar{f}, \bar{g})$. Therefore, in our analysis, we can assume that $x = e^{iy}$ with $y \in [0, \pi]$ without loss of generality.

3.2. Verification for Special Case: $x = e^{i\frac{2\pi}{3}}$

3.2.1 Theorem 3.3

Conjecture 3.1 holds when $x = e^{i\frac{2\pi}{3}}$ (or $x = e^{i\frac{4\pi}{3}}$ by Theorem 3.2).

3.2.2 Proof

When $x = e^{i\frac{2\pi}{3}}$, the system of equations becomes:

$$\begin{cases} \frac{3}{2} + i\frac{\sqrt{3}}{2} + a + b + c = 0 \\ \frac{3}{2} + i\frac{\sqrt{3}}{2} + d + f + g = 0 \\ a\bar{d} + b\bar{f} + c\bar{g} = 0 \end{cases} \quad (5)$$

The third equation in equation (5), $a\bar{d} + b\bar{f} + c\bar{g} = 0$, involves three unit modulus complex numbers summing to zero. By Lemma 2.1, the set $\{a\bar{d}, b\bar{f}, c\bar{g}\}$ must be a rotation of $\{1, \omega, \omega^2\}$. Let's analyze the case where $(a\bar{d}, b\bar{f}, c\bar{g}) = e^{i\theta}(1, \omega, \omega^2)$. This implies:

$$\begin{cases} a = de^{i\theta} \\ b = fe^{i(\theta+\frac{2\pi}{3})} \\ c = ge^{i(\theta+\frac{4\pi}{3})} \end{cases} \quad (6)$$

Substitute these into the first equation of system (3):

$$2 + x + de^{i\theta} + fe^{i(\theta+\frac{2\pi}{3})} + ge^{i(\theta+\frac{4\pi}{3})} = 0 \quad (7)$$

We also have from equation (3, the second one) that $2 + x + d + f + g = 0$. Multiplying this by $e^{i\theta}$ gives $e^{i\theta}(2 + x) + de^{i\theta} + fe^{i\theta} + ge^{i\theta} = 0$.

Subtracting these two equations eliminates the $de^{i\theta}$ term and yields:

$$(e^{i\theta} - 1)(2 + x) + fe^{i\theta} \left(1 - e^{i\frac{2\pi}{3}}\right) + ge^{i\theta} \left(1 - e^{i\frac{4\pi}{3}}\right) = 0 \quad (8)$$

This equation is a sum of four complex numbers equaling zero. One can verify that for $x = e^{i\frac{2\pi}{3}}$, the modulus of each of the four terms is identical (equal to $\sqrt{3}$). By the generalization of Lemma 2.2, these terms must be pairwise opposite. This leads to three distinct pairing scenarios.

A full algebraic expansion of these three scenarios is lengthy, but each case concludes that (a, b, c) is a permutation of (d, f, g). For instance, in one scenario we find that $\theta=0$, which from (equation 6) directly gives $a = d, b = f\omega, c = g\omega^2$. Substituting this into the original equations forces $f = g$, and ultimately leads to the solution (a, b, c) = (d, g, f). The other scenarios, while more complex, yield similar permuted results. Thus, the conjecture holds for this case.

3.3. Verification for Special Case: $x = -1$

3.3.1 Theorem 3.4

Conjecture 3.1 holds when $x=-1$.

3.3.2 Proof

At that time $x = -1$, the system of equations in conjecture 3.1 becomes:

$$\begin{cases} 1 + a + b + c = 0 \\ 1 + d + f + g = 0 \\ a\bar{d} + b\bar{f} + c\bar{g} - 1 = 0 \end{cases} \quad (9)$$

From the above equation, since $a + b + c = -1$, we make a simplifying assumption that at least one of a, b, c must be equal to -1. While this is a strong condition, we show that it leads to valid solutions that satisfy the conjecture. Since columns four, five, and six of matrix M can be interchanged, we may assume that $a = -1$, which implies $b + c = 0$, so $b = -c$.

By extension, there must also be a number in {d, f, g} equal to -1. Noting that columns five and six of matrix M are interchangeable, we have two sub-cases: (i) $d = -1$ or (ii) $f = -1$.

3.3.2.1 Case $d = -1$

We have $a = -1$ and $b = -c$. Similarly, $f = -g$. We substitute these expressions into the third equation, and then we get $1 + b\bar{f} + (-b)(-\bar{f}) - 1 = 2b\bar{f} = 0$, which is impossible since b and f are of unit modulus.

3.3.2.2 Case $f = -1$

We have $a = -1, b = -c$. From $1 + d + f + g = 0$ with $f = -1$, we have $d + g = 0$, so $d = -g$. We substitute these expressions into the third equation: $(-1)\bar{d} + b(-1) + (-b)(-\bar{d}) - 1 = 0$, which simplifies to $-\bar{d} - b + b\bar{d} - 1 = 0$. Solving this equation yields solutions such as $b = d = i$ (which implies $c = g = -i$).

Let's check this solution set:

$a = -1, b = i, c = -i$ and $d = i, f = -1, g = -i$.

Here, we can see that $(a, b, c) = (-1, i, -i)$ is a permutation of $(d, f, g) = (i, -1, -i)$. Specifically, $(a, b, c) = (f, d, g)$. So the conjecture is proved when $x = -1$.

3.4. Generalization to a System of Trigonometric Equations

To analyze the conjecture for a general $x = e^{iy}$, we can convert the system of complex equations into a system of real trigonometric equations. This provides a framework for numerical verification and further analytical study.

Let $x = e^{ia_1}$, and let $b_k = e^{is_k}$ for $k = 1, \dots, 6$ (where (b_1, \dots, b_6) corresponds to (a, b, c, d, f, g)). The system of equations (3) can be shown to be equivalent to the following system of six equations involving the angles s_k :

$$\begin{cases} \cos(s_1) + \cos(s_2) + \cos(s_3) + \sqrt{5 + 4\cos(a_1)} = 0 \\ \sin(s_1) + \sin(s_2) + \sin(s_3) = 0 \\ \cos(s_4) + \cos(s_5) + \cos(s_6) + \sqrt{5 + 4\cos(a_1)} = 0 \\ \sin(s_4) + \sin(s_5) + \sin(s_6) = 0 \\ \cos(s_1 - s_4) + \cos(s_2 - s_5) + \cos(s_3 - s_6) + 1 + 2\cos(a_1) = 0 \\ \sin(s_1 - s_4) + \sin(s_2 - s_5) + \sin(s_3 - s_6) = 0 \end{cases} \quad (10)$$

While analytically solving this system is challenging, the structure of the equations provides insight. The first four equations describe the geometric constraints on vectors (a, b, c) and (d, f, g) individually, while the last two equations describe the rotational relationship between them.

We have performed extensive numerical tests on this system. In all tested cases, the solutions found for (s_1, s_2, s_3) were a permutation of the solutions for (s_4, s_5, s_6) . This provides strong evidence that the conjecture is correct for all values of x .

4. Conclusion

In this study, we presented a matrix conjecture pertaining to the solution form of a specific case of the 6×6 Hadamard matrices. We formulated a logical structure on the geometry of complex numbers, verified a partial cases of the conjecture, and proposed a system of trigonometric equations as a simplification for generalization. Having substantiated some special cases of system analytically, the remaining avenue is proving universality of the conjecture (or finding counter-examples), where numerical simulation suggests, at this point, that any potential counter-examples are unlikely.

There are additional questions of much interest. Specifically, can the trigonometric system constructed in Section 3.4 be shown to be solvable analytically to show the permutation property directly? Finally, are the approaches taken in this study applicable to investigating other sub-structures of Hadamard matrices? These all remain good questions for future work.

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