Cramer's Rule in Hyperbolic Number Plane

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Abstract. This paper investigates the existence theorem for solutions to hyperbolic linear systems and presents the judgment conditions for the existence of such solutions. It extends the judgment conditions applicable to general linear systems, expanding the research scope from real matrices, a focus in advanced algebra, to hyperbolic matrices. The paper derives the form of solutions to hyperbolic linear systems and conducts a more comprehensive discussion, categorizing scenarios into cases of solvability, unsolvability, and uniqueness of solutions. The conclusions drawn in this study can be further applied to research on a broader range of physical problems with concrete physical backgrounds, laying a solid research foundation and injecting new momentum into the application of hyperbolic analysis in the field of algebra.

Keywords: Hyperbolic numbers, system of linear equations, partial order.

1. Introduction

Zengrui Kang and Yuxun Zeng studied the properties of continuous functions on commutative rings with zero divisors generated by two real numbers in the P, proved their boundedness via p-array decomposition with geometric interpretation, and obtained the continuous boundedness theorem for confusing functions [1]. Yuxun Zeng and Zengrui Kang studied vector spaces and linear transformations over the ring P, establishing relevant principles and examining their properties. They proved the existence theorem of linear mappings over P, providing a solid theoretical framework and advancing research in the field [2]. Jinle Hu and Kexin Yan studied arithmetic operation sequences of split complex numbers, designing a sequence convergence separation algorithm for zero divisor problems, which advances their limit theory and provides a basis for physics and other fields [3]. Orhan Eren and Yukel Soykan studied generalized hyperbolic Woodall numbers and their special cases, obtained Binet formulas, etc., and presented Catalan identities and related matrices. [4]. Bryson Kagy and Seth Sullivant studied the polyhedral geometry and isometric properties of distance matrices constructed from circular split systems, presented facets of defining inequalities and so on, and explained the connection with related polytopes [5]. The previous research efforts on functions over commutative rings and linear transformations on rings, among others, have established a theoretical foundation through methods such as decomposition and construction. Their approaches inspire this paper. This paper focuses on the existence theorem for solutions of hyperbolic linear systems, derives the forms of the solutions, extends the judgment conditions to hyperbolic matrices, and categorizes and discusses the solvability, unsolvability, and uniqueness of the solutions. The conclusions can be applied to the research of physical problems and inject new impetus into the application of hyperbolic analysis in the field of algebra.

Aynur Ali, Miroslav Hristov, and others studied the unification of a known technique for fixed points and various types of fixed points (including coupled, triple, and N-tuple fixed points) of weakly monotonic mappings in partially ordered metric spaces. They also explored weakening classical contraction conditions so that they apply only to monotonic continuous iteration sequences generated by the mappings. Their main results encompass several known findings and can be applied to solving nonlinear matrix equations, with similar methods applicable to systems of equations [6]. SK. Safique Ahmad and Neha Bhadala developed a framework for structurally constrained least-squares solutions to generalized reduced biquaternion matrix equations, which can be applied to inverse problems. They provided a solution framework for related issues and included numerical examples to validate the effectiveness of their approach [7]. Ivan Kyrchei and Dijana Mosic explored new characterizations and expressions of weak group (WG) inverses and their duals over quaternion skew fields, focusing on solving matrix equations. They derived novel characterizations, determinant representations, and Cramer's rules, with numerical examples provided to verify the validity of their results [8]. Lei Shi, Qing Wen Wang and others studied the solvability conditions and general solution of the dual generalized commutative quaternion matrix equation $AXB = C$ and obtained its necessary, sufficient solvability conditions and general solution expression, with numerical examples confirming the conclusion's reliability [9]. Dzhaliuk N.S. and Petrychkovych V.M. investigated methods for solving linear two-sided matrix equations, particularly Sylvester-type equations, over various domains. They described the solution structures, extending and generalizing existing results. Their work provided new solution methods, solution structures, as well as conditions for uniqueness and existence, including contributions over specific rings [10]. Scholars like Aynur Ali and Miroslav Hristov have contributed significantly to fixed - point and matrix equation studies, laying theoretical foundations. Their work inspires ours. As in our abstract, we explore hyperbolic linear system solution existence, extend scope to hyperbolic matrices, derive forms, and categorize scenarios. Drawing on predecessors'methods, we aim to solve complex constraints and innovate in matrix - related equation theory and application.

ElDahshan A K and others studied dynamic motion prediction, proposed elliptical/ellipsoidal PDF models with velocity-direction, combined with Markov models to improve accuracy and introduced applications [11]. Manoharan Kesavan and others studied construction supervision features with mathematical application, formed validated competency guidelines to help address industry knowledge gaps [12]. Yasir Ramzan and Hanadi Alzubadi and others studied Lassa fever transmission dynamics and neurological impacts, modeled parameters/interventions, identified associated disabilities and optimized prevention directions [13]. Haneen Hamam and Yasir Ramzan and others studied a mathematical framework for Lassa virus transmission among heterosexuals, modeled parameters/interventions, and formulated and evaluated transmission control strategies [14]. Alam Sabrina Shajeen and Dube Adam Kenneth studied the development of a digital home numeracy practice inventory; validated its 19-item, 5-construct model as reliable and effective to explore impacts on young children's math development [15]. These rich and diverse studies, ranging from model construction for dynamic motion prediction, to exploration of mathematical frameworks for disease transmission, and to research on digital practice in the education field, have established solid practical and theoretical foundations for different disciplinary directions. Their ideas of cross - domain model application, rigorous parameter modeling and verification methods inspire us in our research to draw on the mathematical modeling logic in multiple scenarios, break through the limitations of a single domain, and attempt to integrate multiple methods to solve complex problems, exploring new paths for the deepened application of relevant theories in practical scenarios.

Building on the work of previous researchers, we further discuss the relationship between the coefficient matrix and solutions of linear systems on the hyperbolic number plane and use the rank of the matrix to determine the existence of solutions for hyperbolic linear systems.

2. Preliminaries

The set $\mathbb{D}$ defined as $\mathbb{D} := \{z = x + \mathbf{k}y : x, y \in \mathbb{R}\}$ where the hyperbolic unit $\mathbf{k}$ satisfies $\mathbf{k}^2 = 1$ and $\mathbf{k} \notin \mathbb{R}$ is a commutative ring equipped with the operations of addition and multiplication.

For any two hyperbolic numbers $z_1 = x_1 + \mathbf{k}y_1$ and $z_2 = x_2 + \mathbf{k}y_2 \in \mathbb{D}$, the addition is given by:

$$z_1 + z_2 = (x_1 + x_2) + \mathbf{k}(y_1 + y_2), \tag{1}$$

and the multiplication is defined as:

$$\begin{aligned} z_1 z_2 &= (x_1 + \mathbf{k}y_1)(x_2 + \mathbf{k}y_2) \\ &= (x_1x_2 + y_1y_2) + \mathbf{k}(x_1y_2 + x_2y_1). \end{aligned} \tag{2}$$

For any hyperbolic number $z = x + \mathbf{k}y \in \mathbb{D}$, the real part is defined as $\mathrm{Re}(z) = x$ and the hyperbolic part as $\mathrm{Im}(z) = y$. The conjugate of z. is denoted by $\bar{z} = x - \mathbf{k}y$. The ring $\mathbb{D}$ contains zero-divisors.

The hyperbolic numbers can also be expressed in terms of the idempotent basis $\{\mathbf{e}, \mathbf{e}^\dagger\}$, where:

$$\mathbf{e} := \frac{1+\mathbf{k}}{2} \quad \text{and} \quad \mathbf{e}^\dagger := \frac{1-\mathbf{k}}{2}, \tag{3}$$

and

$$\mathbf{e}\mathbf{e}^\dagger = 0. \tag{4}$$

These elements satisfy the idempotent properties:

$$\mathbf{e} \times \mathbf{e} = \mathbf{e}, \quad \mathbf{e}^\dagger \times \mathbf{e}^\dagger = \mathbf{e}^\dagger, \tag{5}$$

and the relations:

$$\mathbf{e} + \mathbf{e}^\dagger = 1, \quad \mathbf{e} - \mathbf{e}^\dagger = \mathbf{k}. \tag{6}$$

Any hyperbolic number $z = x + \mathbf{k}y \in \mathbb{D}$ can be represented in the idempotent basis as:

$$\begin{aligned} z &= (x+y)\mathbf{e} + (x-y)\mathbf{e}^\dagger \\ &=: a\mathbf{e} + b\mathbf{e}^\dagger, \end{aligned} \tag{7}$$

where $a = x + y$ and $b = x - y$. The zero-divisors in $\mathbb{D}$ are real multiples of $\mathbf{e}$ or $\mathbf{e}^\dagger$, corresponding to the lines $y = x$ and $y = -x$ in the hyperbolic plane. Specifically, a hyperbolic number z is a zero-divisor if and only if $z = x\mathbf{e}$ or $z = y\mathbf{e}^\dagger$ for $x, y \in \mathbb{R} \setminus \{0\}$.

For hyperbolic numbers $z_1 = a_1\mathbf{e} + a_2\mathbf{e}^\dagger$ and $z_2 = b_1\mathbf{e} + b_2\mathbf{e}^\dagger$, the addition and multiplication are given by:

$$z_1 + z_2 = (a_1 + b_1)\mathbf{e} + (a_2 + b_2)\mathbf{e}^\dagger, \tag{8}$$

$$z_1 z_2 = (a_1b_1)\mathbf{e} + (a_2b_2)\mathbf{e}^\dagger. \tag{9}$$

The set of non-negative hyperbolic numbers is defined as:

$$\mathbb{D}^+ = \{z = a\mathbf{e} + b\mathbf{e}^\dagger : a \ge 0, b \ge 0\}, \tag{10}$$

and the set of non-positive hyperbolic numbers is defined as:

$$\mathbb{D}^- = \{z = a\mathbf{e} + b\mathbf{e}^\dagger : a \le 0, b \le 0\}. \tag{11}$$

We observe that if $z_1 = x_1 + \mathbf{k}y_1$, then $z_1 z_2 \in D^+$. Similarly, if $z_1, z_2 \in D^-$, then $z_1 z_2 \in D^-$. Consequently, D^+ is closed under the multiplication operation.

For any $z, w \in D$, if we have:

$$z \preceq w \text{ if and only if } w - z \in D^+ \tag{12}$$

then we say that w is D-greater than z in D. Also, if we have

$$w \preceq z \text{ if and only if } w - z \in D^- \tag{13}$$

Then we say w is D-less than z in D.

Additionally, $z \in D^+$ is equivalent to $z \succeq 0$, while $z \in D^+ \setminus \{0\}$ corresponds to $z \succ 0$. Conversely, $z \in D^-$ is equivalent to $z \preceq 0$. and $z \in D^- \setminus \{0\}$ corresponds to $z \prec 0$. For $z_1 = a_1\mathbf{e} + a_2\mathbf{e}^\dagger$ and $z_2 = b_1\mathbf{e} + b_2\mathbf{e}^\dagger$, we say $z_1 \preceq z_2$, if and only if $a_1 \le b_1$ and $a_2 \le b_2$. This relation $\preceq$ is reflexive, transitive, and antisymmetric, thus it represents a partial order in D. This order can be observed in Figure 1, where the x-axis and y-axis denote a one-dimensional representation.

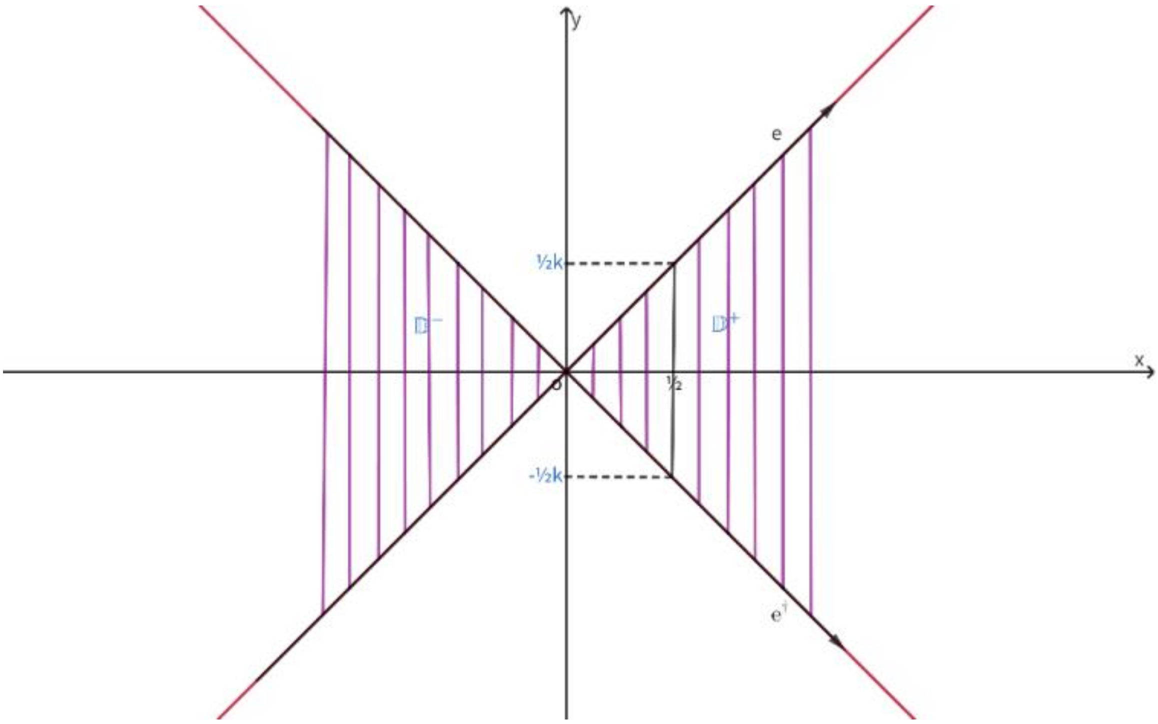

Figure 1. The concept of positive and negative hyperbolic numbers

The closed hyperbolic interval $[z, w]_D$ is defined by

$$[z, w]_D = \{u \in D : z \preceq u \preceq w\}. \tag{14}$$

Equivalently, $u = u_1\mathbf{e} + u_2\mathbf{e}^\dagger \in [z, w]_D$. if and only if

$$a_1 \le u_1 \le a_2 \quad \text{and} \quad b_1 \le u_2 \le b_2, \tag{15}$$

where $z = a_1\mathbf{e} + b_1\mathbf{e}^\dagger$ and $w = a_2\mathbf{e} + b_2\mathbf{e}^\dagger$ in D.

Take now $a = \mathbf{k}$ and $b = 2$. In this case the hyperbolic interval is given by

$$[\mathbf{k}, 2]_D = \left\{ z = \beta_1\mathbf{e} + \beta_2\mathbf{e}^\dagger : 1 \le \beta_1 \le 2 \text{ and } -1 \le \beta_2 \le 2 \right\}, \tag{16}$$

and it is now a two-dimensional set. See Figure 2.

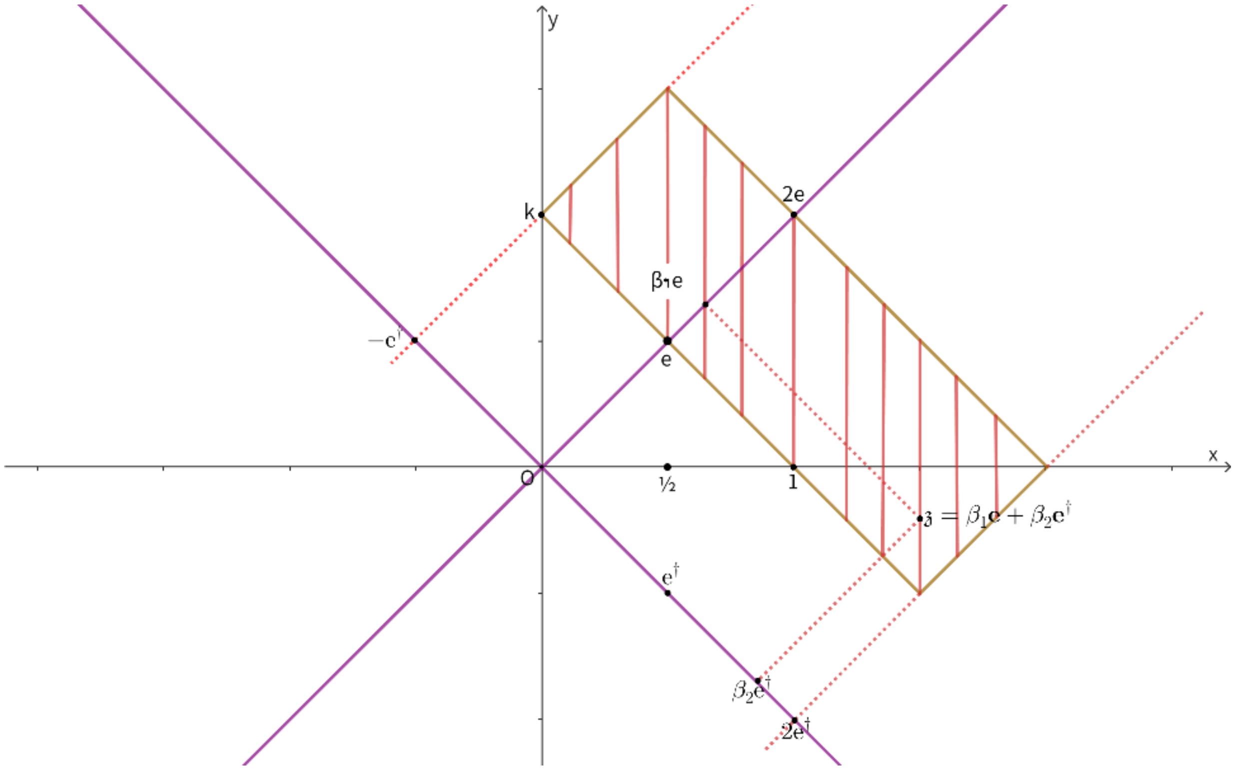

Figure 2. The hyperbolic interval $[\mathbf{k}, 2]_{\mathrm{D}}$

This poset diagram is used to explain the algebraic structure of the set of hyperbolic numbers, with the core focusing on the idempotent decomposition of hyperbolic numbers. Based on a plane coordinate system, the diagram constructs an oblique basis vector system by means of the idempotents $\mathbf{e}$ and $\mathbf{e}^{\dagger}$ of hyperbolic numbers. A hyperbolic number z can be expressed as $z = \beta_1\mathbf{e} + \beta_2\mathbf{e}^{\dagger}$, where β_1 and β_2 are random values within the corresponding restricted intervals. The shaded parallelogram in the diagram intuitively presents the component $\beta_1\mathbf{e}$ of this hyperbolic number along the $\mathbf{e}$-direction and the component $\beta_2\mathbf{e}^{\dagger}$ along the $\mathbf{e}^{\dagger}$-direction. It not only reflects the geometric form of the hyperbolic number based on idempotent decomposition but also intuitively explains the concept of poset intervals, assisting in understanding its algebraic structure. For example, $\mathbf{e}$, $\mathbf{e}^{\dagger}$ and their scalar multiplications demonstrate the scaling of basis vectors. Overall, it transforms the algebraic decomposition of hyperbolic numbers into a geometrically intuitive parallelogram projection, helping to comprehend its structural characteristics under a similar "two-dimensional decomposition".

2.1. Norm on the hyperbolic number set

Any hyperbolic number $\mathrm{z} = x + \mathbf{k}y = a\mathbf{e} + b\mathbf{e}^{\dagger}$ can be represented by the point $(x, y) \in \mathbb{R}^2$ or the point $(a, b) \in \mathbb{R}^2$, respectively. The modulus of hyperbolic numbers can be defined based on the standard base or the idempotent base. Literature indicates that the modulus of a hyperbolic number can be expressed in two ways: one as a positive hyperbolic number, and the other as a positive real number. In this article, the latter definition is preferred. Specifically, the norm used here is based on the standard base of hyperbolic numbers and aligns with the Lorentz scalar product.

For a given hyperbolic number $\mathrm{z} = a\mathbf{e} + b\mathbf{e}^{\dagger}$, the positive hyperbolic number

$$|\,\mathrm{z}|_{\mathrm{k}} = |\,a\mathbf{e} + b\mathbf{e}^{\dagger}\,|_{\mathrm{k}} = |\,a\,|\,\mathbf{e} + |\,b\,|\,\mathbf{e}^{\dagger} \tag{17}$$

is referred to as the hyperbolic modulus of z.

$|z|_k$ serves as a modulus, satisfying the following properties for any $z, w \in \mathbb{D}$:

- $|z|_k = 0$ if and only if $z = 0$,
- $|zw|_k = |z|_k |w|_k$,
- $|z+w|_k \leq |z|_k + |w|_k$.

The relationship between the Euclidean norm $|z|$ and the hyperbolic value norm $|z|_k$ is given by

$$||z|_k| = \frac{1}{2}\sqrt{a^2+b^2} = |z|, \tag{18}$$

where a and b are the coefficients of z.

2.2. Matrix representations of hyperbolic numbers

Consider $\mathbb{R}^{m\times n}$ as the collection of all $m\times n$ real matrices, which is recognized as a vector space equipped with standard matrix addition and scalar multiplication over $\mathbb{R}$. Let $\mathbb{H}^{m\times n}$ denote the set of $m\times n$ matrices whose elements are hyperbolic numbers. A hyperbolic matrix can be expressed in the following manner:

$$\begin{aligned} \mathcal{A} &= \begin{bmatrix} a_{11}\mathbf{e}+b_{11}\mathbf{e}^\dagger & a_{12}\mathbf{e}+b_{12}\mathbf{e}^\dagger & \cdots & a_{1n}\mathbf{e}+b_{1n}\mathbf{e}^\dagger \\ a_{21}\mathbf{e}+b_{21}\mathbf{e}^\dagger & a_{22}\mathbf{e}+b_{22}\mathbf{e}^\dagger & \cdots & a_{2n}\mathbf{e}+b_{2n}\mathbf{e}^\dagger \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}\mathbf{e}+b_{m1}\mathbf{e}^\dagger & a_{m2}\mathbf{e}+b_{m2}\mathbf{e}^\dagger & \cdots & a_{mn}\mathbf{e}+b_{mn}\mathbf{e}^\dagger \end{bmatrix} \\ &= \begin{bmatrix} a_{11}\mathbf{e} & a_{12}\mathbf{e} & \cdots & a_{1n}\mathbf{e} \\ a_{21}\mathbf{e} & a_{22}\mathbf{e} & \cdots & a_{2n}\mathbf{e} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}\mathbf{e} & a_{m2}\mathbf{e} & \cdots & a_{mn}\mathbf{e} \end{bmatrix} + \begin{bmatrix} b_{11}\mathbf{e}^\dagger & b_{12}\mathbf{e}^\dagger & \cdots & b_{1n}\mathbf{e}^\dagger \\ b_{21}\mathbf{e}^\dagger & b_{22}\mathbf{e}^\dagger & \cdots & b_{2n}\mathbf{e}^\dagger \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1}\mathbf{e}^\dagger & b_{m2}\mathbf{e}^\dagger & \cdots & b_{mn}\mathbf{e}^\dagger \end{bmatrix} \\ &= A\mathbf{e} + B\mathbf{e}^\dagger \end{aligned} \tag{19}$$

where A and B represent $m\times n$ real matrices. It is noteworthy that each hyperbolic matrix can be formulated as a linear combination of two real matrices. Consequently, the set of hyperbolic matrices can be defined as

$$\mathbb{H} = \{\mathcal{A} = A\mathbf{e} + B\mathbf{e}^\dagger : A, B \in \mathbb{R}^{m\times n}, \mathbf{k}^2 = 1, \mathbf{k} \notin \mathbb{R}\}. \tag{20}$$

Several distinct categories of hyperbolic matrices are identified based on their structure and properties:

- **Column/Row Matrix:** Matrices that include just one column (line) are called column (line) hyperbolic matrices.
- **Zero Matrix:** If all entries of the matrix are 0, then it is called a zero hyperbolic matrix and is shown as $0_{\mathbb{H}}$.
- **Square Matrix:** If the number of rows equals the number of columns of a hyperbolic matrix, that is $n\times n$ type, then the matrix is called a square hyperbolic matrix.
- **Unit Matrix:** For square hyperbolic matrices in which all the entries on the diagonal are equal to 1 and others are 0, called unit hyperbolic matrices and denoted by $I_{\mathbb{H}}$. It can be easily seen that the unit hyperbolic matrix and the unit real matrix are equal. Moreover, for all $\mathcal{A} \in \mathbb{H}^n_{\mathbb{H}}$, $I_{\mathbb{H}} \cdot \mathcal{A} = \mathcal{A} \cdot I_{\mathbb{H}} = \mathcal{A}$.

Let $A = [\mathbf{e}a_{ij} + \mathbf{e}^\dagger a^*_{ij}]$ and $B = [\mathbf{e}b_{ij} + \mathbf{e}^\dagger b^*_{ij}]$ be two $m\times n$ hyperbolic matrices and $\lambda \in \mathbb{H}$ be a scalar.

- **Equality:** If $a_{ij} = b_{ij}$ and $a_{ij}^* = b_{ij}^*$ for all i and j, then these hyperbolic matrices are equal.
- **Addition:** $A+B = [\mathbf{e}(a_{ij} + b_{ij}) + \mathbf{e}^\dagger (a_{ij}^* + b_{ij}^*)]$.
- **Scalar Multiplication:** $\lambda A = [\lambda(\mathbf{e}a_{ij} + \mathbf{e}^\dagger a_{ij}^*)]$.

2.3. Rank of hyperbolic matrices

Consider a set of vectors $z_1, z_2, \ldots, z_j$, where each z_i is a vector with entries from the hyperbolic numbers $\mathrm{D} = \{x_0 + x_1\mathbf{k} : x_0, x_1 \in \mathbb{R}, \mathbf{k}^2 = 1, \mathbf{k} \notin \mathbb{R}\}$. These vectors are classified as linearly dependent if there exists a nontrivial linear relation among them. Specifically, there must be a relation of the form

$$k_1 z_1 + \cdots + k_j z_j = \mathbf{0}, \tag{21}$$

where the coefficients $k_1, \ldots, k_j$ are hyperbolic numbers (i.e., $k_i \in \mathrm{D}$), and not all $k_1, \ldots, k_j$ are equal to the zero hyperbolic number $\mathbf{e}$ or $\mathbf{e}^\dagger$.

A set of vectors $z_1, z_2, \ldots, z_j$ with entries in D is termed linearly independent if it is not linearly dependent.

Referring to Definition 2.12 in the work of Ferhat Kuruz and Ali Dagdeviren, we adopt their definition of the trace of a hyperbolic matrix [16]. Furthermore, based on the factorization property of zero-divisors in hyperbolic numbers, we introduce the following definition of the rank of a hyperbolic matrix. For a matrix $\mathcal{A} \in \mathrm{D}^{m\times n}$, where the entries are expressed as hyperbolic numbers, the rank of $\mathcal{A}$, denoted $\mathrm{rank}(\mathcal{A})$, is defined as

$$\mathrm{rank}(\mathcal{A}) = \mathrm{rank}(A)\mathbf{e} + \mathrm{rank}(B)\mathbf{e}^\dagger, \tag{22}$$

where $\mathrm{rank}(A)$ is the rank of A and $\mathrm{rank}(B)$ is the rank of B. This is a mapping from $\mathrm{D}^{m\times n}$ to D^+. This reflects the maximum number of linearly independent rows or columns, corresponding to the dimensional contribution of the $\mathbf{e}$-component (row or column space spanned by $A\mathbf{e}$) and the $\mathbf{e}^\dagger$-component (row or column space spanned by $B\mathbf{e}^\dagger$) within D^m or D^n.

A matrix $\mathcal{A}$ has full rank if A and B have full rank. More special, $\mathrm{rank}(\mathcal{A}) = n$ if $\mathcal{A}$ is a square matrix.

3. Rank of hyperbolic matrices and solution of System of Complex Equations

Theorem 1 (Hyperbolic Cramer's Rule) Let $\mathcal{A}$ be an $n \times n$ matrix with entries from the hyperbolic numbers $\mathrm{D} = \{x_0 + x_1\mathbf{k} : x_0, x_1 \in \mathbb{R}, \mathbf{k}^2 = 1, \mathbf{k} \notin \mathbb{R}\}$, and let $\hat{b}$ be an $m \times 1$ column vector with entries also from D. Consider the hyperbolic linear equation system $\mathcal{A}x = \hat{b}$. Let

$$r = \mathrm{rank}(\mathcal{A}) = \mathrm{rank}(A)\mathbf{e} + \mathrm{rank}(B)\mathbf{e}^\dagger = r_1\mathbf{e} + r_2\mathbf{e}^\dagger \in \mathrm{D}^+ \tag{23}$$

denote the rank of $\mathcal{A}$ and

$$r' = \mathrm{rank}([\mathcal{A} \mid \hat{b}]) = \mathrm{rank}([A \mid \hat{b}])\mathbf{e} + \mathrm{rank}([B \mid \hat{b}])\mathbf{e}^\dagger = r_1'\mathbf{e} + r_2'\mathbf{e}^\dagger \in \mathrm{D}^+ \tag{24}$$

denote the rank of the augmented matrix $[\mathcal{A} \mid \hat{b}]$, where $[\mathcal{A} \mid \hat{b}]$ is the $m \times (n+1)$ matrix formed by horizontally concatenating $\mathcal{A}$ and $\hat{b}$. Then:

I. If any of the conditions $r_1' > r_1$, $r_2' > r_2$, or $r' \succ r$ holds, the hyperbolic equation system $\mathcal{A}x = \hat{b}$ has no solution. This indicates that the rank increase in the augmented matrix suggests $\hat{b}$ cannot be expressed as a linear combination of the columns of $\mathcal{A}$.

II. If $r' = r$, the hyperbolic equation system $\mathcal{A}x = \hat{b}$ has a solution.

  i If $r' = r = n$, the equation system has a unique solution. This corresponds to $\mathcal{A}$ being a full-rank square matrix, where its columns are linearly independent in $\mathbb{D}^m$, uniquely determining $\hat{b}$.

  ii If any of the conditions $r_1 < n$ and $r_2 = n$, $r_2 < n$ and $r_1 = n$, or $r \prec n$ holds, the equation system has infinitely many solutions. This reflects that the columns of $\mathcal{A}$ do not fully constrain the solution, allowing free parameters in $\mathbb{D}$.

Proof We consider the linear system $\mathcal{A}x = \hat{b}$. The matrix $\mathcal{A}$ is expressed as:

$$\begin{aligned}
\mathcal{A} &= \begin{bmatrix} a_{11}\mathbf{e} + b_{11}\mathbf{e}^\dagger & a_{12}\mathbf{e} + b_{12}\mathbf{e}^\dagger & \cdots & a_{1n}\mathbf{e} + b_{1n}\mathbf{e}^\dagger \\ a_{21}\mathbf{e} + b_{21}\mathbf{e}^\dagger & a_{22}\mathbf{e} + b_{22}\mathbf{e}^\dagger & \cdots & a_{2n}\mathbf{e} + b_{2n}\mathbf{e}^\dagger \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}\mathbf{e} + b_{m1}\mathbf{e}^\dagger & a_{m2}\mathbf{e} + b_{m2}\mathbf{e}^\dagger & \cdots & a_{mn}\mathbf{e} + b_{mn}\mathbf{e}^\dagger \end{bmatrix} \\
&= \begin{bmatrix} a_{11}\mathbf{e} & a_{12}\mathbf{e} & \cdots & a_{1n}\mathbf{e} \\ a_{21}\mathbf{e} & a_{22}\mathbf{e} & \cdots & a_{2n}\mathbf{e} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}\mathbf{e} & a_{m2}\mathbf{e} & \cdots & a_{mn}\mathbf{e} \end{bmatrix} + \begin{bmatrix} b_{11}\mathbf{e}^\dagger & b_{12}\mathbf{e}^\dagger & \cdots & b_{1n}\mathbf{e}^\dagger \\ b_{21}\mathbf{e}^\dagger & b_{22}\mathbf{e}^\dagger & \cdots & b_{2n}\mathbf{e}^\dagger \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1}\mathbf{e}^\dagger & b_{m2}\mathbf{e}^\dagger & \cdots & b_{mn}\mathbf{e}^\dagger \end{bmatrix} \\
&= A\mathbf{e} + B\mathbf{e}^\dagger
\end{aligned} \tag{25}$$

where A and B are real coefficient matrices. The unknown vector x is expressed as:

$$x = \begin{pmatrix} x_{11}\mathbf{e} + x_{12}\mathbf{e}^\dagger \\ x_{21}\mathbf{e} + x_{22}\mathbf{e}^\dagger \\ \vdots \\ x_{n1}\mathbf{e} + x_{n2}\mathbf{e}^\dagger \end{pmatrix} = \begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{pmatrix}\mathbf{e} + \begin{pmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{n2} \end{pmatrix}\mathbf{e}^\dagger = x_1\mathbf{e} + x_2\mathbf{e}^\dagger, \tag{26}$$

where x_1 and x_2 are real vectors. The right-hand side vector $\hat{b}$ is expressed as:

$$\hat{b} = \begin{pmatrix} \hat{b}_{11}\mathbf{e} + \hat{b}_{12}\mathbf{e}^\dagger \\ \hat{b}_{21}\mathbf{e} + \hat{b}_{22}\mathbf{e}^\dagger \\ \vdots \\ \hat{b}_{n1}\mathbf{e} + \hat{b}_{n2}\mathbf{e}^\dagger \end{pmatrix} = \begin{pmatrix} \hat{b}_{11} \\ \hat{b}_{21} \\ \vdots \\ \hat{b}_{n1} \end{pmatrix}\mathbf{e} + \begin{pmatrix} \hat{b}_{12} \\ \hat{b}_{22} \\ \vdots \\ \hat{b}_{n2} \end{pmatrix}\mathbf{e}^\dagger = \hat{b}_1\mathbf{e} + \hat{b}_2\mathbf{e}^\dagger, \tag{27}$$

where $\hat{b}_1$ and $\hat{b}_2$ are real vectors.

Substituting these into the equation $\mathcal{A}x = \hat{b}$, we obtain:

$$(A\mathbf{e} + B\mathbf{e}^\dagger)(x_1\mathbf{e} + x_2\mathbf{e}^\dagger) = \hat{b}_1\mathbf{e} + \hat{b}_2\mathbf{e}^\dagger. \tag{28}$$

Expanding the left-hand side:

$$Ax_1\mathbf{e} + Bx_2\mathbf{e}^\dagger = \hat{b}_1\mathbf{e} + \hat{b}_2\mathbf{e}^\dagger. \tag{29}$$

Since $\mathbf{e}$ and $\mathbf{e}^{\dagger}$ form a pair of linearly independent bases, the equation separates into two independent real linear systems:

$$Ax_1 = \hat{b}_1, \quad Bx_2 = \hat{b}_2. \tag{30}$$

Explicitly, we have:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{pmatrix} = \begin{pmatrix} \hat{b}_{11} \\ \hat{b}_{21} \\ \vdots \\ \hat{b}_{m1} \end{pmatrix}, \quad \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix} \begin{pmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{n2} \end{pmatrix} = \begin{pmatrix} \hat{b}_{12} \\ \hat{b}_{22} \\ \vdots \\ \hat{b}_{m2} \end{pmatrix}. \tag{31}$$

We now discuss the existence and uniqueness of solutions:

I. If $r_1' > r_1$, the subsystem $Ax_1 = \hat{b}_1$ has no solution, implying that the entire system $\mathrm{A}x = \hat{b}$ has no solution. Similarly, if $r_2' > r_2$, the system also has no solution. Specifically, if $r_1' > r_1$ and $r_2' > r_2$, i.e., $r' \succ r$, the system has no solution.

II. If $r_1' = r_1$ and $r_2' = r_2$, i.e., $r' = r$, then both subsystems have solutions, thus $\mathrm{A}x = \hat{b}$ is solvable.

 i If $r_1 = n$ and $r_2 = n$, both subsystems have unique solutions, yielding a unique solution for the entire system:

$$x = x_1'\mathbf{e} + x_2'\mathbf{e}^{\dagger}, \tag{32}$$

 where x_1' and x_2' are solutions to the two subsystems.

 ii If $r_1 < n$ or $r_2 < n$, the corresponding subsystem has infinitely many solutions:

  1) If $r_1 < n$ and $r_2 = n$, the subsystem $Ax_1 = \hat{b}_1$ has infinitely many solutions, hence the entire system $\mathrm{A}x = \hat{b}$ has infinitely many solutions.

  2) If $r_2 < n$ and $r_1 = n$, the subsystem $Bx_2 = \hat{b}_2$ has infinitely many solutions, thus $\mathrm{A}x = \hat{b}$ has infinitely many solutions.

  3) In particular, if $r_1 < n$ and $r_2 < n$, i.e., $r \prec n$ both subsystems have infinitely many solutions, so the entire system $\mathrm{A}x = \hat{b}$ also has infinitely many solutions.

4. Conclusions

This research delves into the existence theorem for solutions to hyperbolic linear systems and formulates key judgment conditions for their existence. By extending the solution criteria applicable to general linear systems, it broadens the research horizon from real matrices—a central focus in advanced algebra—to the domain of hyperbolic matrices. The study not only derives the specific form of solutions for hyperbolic linear systems but also conducts an in-depth analysis, categorizing scenarios into distinct cases of solvability, unsolvability, and uniqueness of solutions. Moreover, it achieves a significant breakthrough by extending Cramer's Rule from traditional linear algebra to the hyperbolic setting, presenting a hyperbolic version of this fundamental rule. These findings lay a robust foundation for applying hyperbolic analysis in algebraic research and offer valuable insights for investigating a wider range of physical problems with concrete backgrounds.

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